

## ON EINSTEIN METRICS

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Let us consider a compact orientable  $C^\infty$  manifold  $M$ . If a  $C^\infty$  Riemannian metric  $g$  is given on  $M$ , we get a Riemannian manifold  $(M, g)$ . Let us consider the set of Riemannian metrics  $g$  on  $M$  such that

$$\int_M dV = 1,$$

where  $dV$  is the volume element of  $(M, g)$ . We denote this set by  $\mathcal{M}(M)$ .

Consider the integral

$$I(g) = \int_M K dV,$$

where  $K$  is the scalar curvature of  $(M, g)$ . It is well known that a critical point  $\bar{g}$  of  $I(g)$  in  $\mathcal{M}(M)$  is an Einstein metric. A question then arises that whether a given Einstein metric gives a minimum of  $I(g)$  or not. It has been established by M. Berger [1] that there exists some Einstein metric (on some  $C^\infty$  manifold) for which both  $I(g)$  and  $-I(g)$  have nonfinite indices.

The purpose of the present paper is to prove the following main theorem.

**Theorem.** *Let  $I(g)$  be the integral as defined above. Then the index of  $I(g)$  and also the index of  $-I(g)$  are positive at each critical point.*

In the last paragraph some suggestion is given about the index.

### 1. Infinitesimal deformations of a Riemannian metric from an Einstein metric

In the following we use local coordinates, and a tensor is expressed in its components with respect to the natural frame. Thus  $K_{kji}{}^h$  means the curvature tensor

$$K_{kji}{}^h = \partial_k \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} + \left\{ \begin{matrix} h \\ kl \end{matrix} \right\} \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jl \end{matrix} \right\} \left\{ \begin{matrix} l \\ ki \end{matrix} \right\},$$

where  $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$  is the Christoffel symbol of the metric  $g$ . The Ricci tensor and the scalar curvature are respectively given by

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$$K_{ji} = K_{tji}{}^t, \quad K = K_{ji}g^{ji} = K^{ji}g_{ji}.$$

When we take a  $C^\infty$  curve  $g(t)$  in  $\mathcal{M}(M)$ , we get several tensor fields defined by

$$D_{ji} = \frac{\partial}{\partial t} g_{ji}, \quad D_i{}^h = D_{ik}g^{kh}, \quad D^{ih} = D_{kj}g^{ki}g^{jh},$$

$$D_{ji}{}^h = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji}), \quad D_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h,$$

where  $\nabla$  means the covariant differentiation with respect to the metric  $g(t)$ . Then we get

$$\frac{\partial}{\partial t} K_{kji}{}^h = D_{kji}{}^h,$$

and consequently

$$\frac{d}{dt} I(g(t)) = \int_M \left[ D_{kji}{}^k g^{ji} - K^{ji} D_{ji} + \frac{1}{2} K g^{ji} D_{ji} \right] dV.$$

Since any divergence vanishes by integration, we immediately obtain

$$(1.1) \quad \frac{d}{dt} I(g(t)) = \int_M \left( -K^{ji} + \frac{1}{2} K g^{ji} \right) D_{ji} dV.$$

On the other hand  $D_{ji}$  must satisfy the only condition

$$(1.2) \quad \int_M g^{ji} D_{ji} dV = 0.$$

Let  $\bar{g}$  be a Riemannian metric such that, for every  $C^\infty$  curve  $g(t)$  of  $\mathcal{M}(M)$  satisfying  $g(0) = \bar{g}$ ,  $I(g(t))$  has vanishing derivative at  $t = 0$ . Then from (1.1) and (1.2) we get

$$K^{ji} = \frac{1}{2} K g^{ji} + c g^{ji}$$

for this metric  $\bar{g}$ . Here  $c$  is a constant, and this leads to the well-known formula of an Einstein space  $(M, \bar{g})$ , namely,

$$(1.3) \quad K_{ji} = \frac{K}{n} g_{ji}.$$

Now let us assume that  $g(0)$  is an Einstein metric and study the behavior of  $d^2 I / dt^2$  at  $t = 0$  for various curves  $g(t)$ .

Differentiating (1.1) again we get

$$\begin{aligned}
 \frac{d^2 I(g(t))}{dt^2} = \int_M \left[ \left( -g^{lj} g^{ki} \frac{\partial K_{lk}}{\partial t} + 2g^{lj} \frac{\partial g_{lk}}{\partial t} K^{ki} \right) \frac{\partial g_{ji}}{\partial t} - K^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} \right. \\
 + \frac{1}{2} \frac{\partial K_{ji}}{\partial t} g^{ji} g^{lk} \frac{\partial g_{lk}}{\partial t} - \frac{1}{2} K^{ji} \frac{\partial g_{ji}}{\partial t} g^{lk} \frac{\partial g_{lk}}{\partial t} \\
 - \frac{1}{2} K g^{lj} g^{ki} \frac{\partial g_{lk}}{\partial t} \frac{\partial g_{ji}}{\partial t} + \frac{1}{2} K g^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} \\
 \left. + \left( -K^{ji} \frac{\partial g_{ji}}{\partial t} + \frac{1}{2} K g^{ji} \frac{\partial g_{ji}}{\partial t} \right) \frac{1}{2} g^{lk} \frac{\partial g_{lk}}{\partial t} \right] dV,
 \end{aligned}
 \tag{1.4}$$

in which we can put

$$\frac{\partial K_{ji}}{\partial t} = \nabla_t D_{ji}{}^t - \nabla_j D_{ti}{}^t.$$

Using Green's theorem we can write (1.4) in the form

$$\begin{aligned}
 \frac{d^2 I}{dt^2} = \int_M \left[ D^{jih} \nabla_h D_{ji} - D_i{}^{it} \nabla^j D_{ji} + 2D_i{}^j K^{it} D_{ji} \right. \\
 - \frac{1}{2} D_s{}^{st} \nabla_t D_i{}^i + \frac{1}{2} D_{st}{}^t \nabla^s D_i{}^i - K^{ji} D_{ji} D_t{}^t \\
 \left. - \frac{1}{2} K D_{ji} D^{ji} + \frac{1}{4} K (D_t{}^t)^2 - \left( K^{ji} - \frac{1}{2} K g^{ji} \right) \frac{\partial^2 g_{ji}}{\partial t^2} \right] dV.
 \end{aligned}$$

Since  $g(0)$  is an Einstein point<sup>1</sup>, we get

$$\int_M \left[ \left( K^{ji} - \frac{1}{2} K g^{ji} \right) \frac{\partial^2 g_{ji}}{\partial t^2} \right] dV = - \left( \frac{1}{2} - \frac{1}{n} \right) K \int_M g^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} dV$$

at  $t = 0$ . On the other hand, differentiating  $\int_M dV = 1$  we get

$$\begin{aligned}
 \int_M g^{ji} \frac{\partial g_{ji}}{\partial t} dV = 0, \\
 \int_M \left[ g^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} - g^{lj} g^{ki} \frac{\partial g_{lk}}{\partial t} \frac{\partial g_{ji}}{\partial t} + \frac{1}{2} \left( g^{ji} \frac{\partial g_{ji}}{\partial t} \right)^2 \right] dV = 0.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \left( \frac{d^2 I}{dt^2} \right)_0 = \int_M \left[ (\nabla^j D^{ih}) \nabla_h D_{ji} - \frac{1}{2} (\nabla^h D^{ji}) \nabla_h D_{ji} - (\nabla^i D_j{}^j) \nabla^h D_{hi} \right. \\
 \left. + \frac{1}{2} (\nabla^i D_j{}^j) \nabla_i D_h{}^h + \frac{K}{n} \left\{ D_{ji} D^{ji} - \frac{1}{2} (D_i{}^i)^2 \right\} \right] dV
 \end{aligned}
 \tag{1.5}$$

<sup>1</sup>We call an Einstein metric an Einstein point in  $\mathcal{M}(M)$ .

for any symmetric tensor field  $D_{ji}$  satisfying (1.2) if  $g(0)$  is an Einstein point.

## 2. Infinitesimal conformal deformations of a Riemannian metric

If we put  $D_{ji} = fg_{ji}$  where  $f$  is a  $C^\infty$  function such that  $\int_M f dV = 0$ , then

$$\left(\frac{d^2I}{dt^2}\right)_0 = \frac{n-2}{2} \left[ (n-1) \int_M (\nabla^i f) \nabla_j f dV - K \int_M f^2 dV \right].$$

There exist functions  $f$  which make this quantity positive. This proves that the index of  $-I(g)$  is positive.

## 3. Infinitesimal deformations of a Riemannian metric in a small neighborhood

Let  $U$  be a coordinate neighborhood of  $M$ , and let  $N \subset U$  be a neighborhood of a point  $P_0 \in U$ , where the local coordinates are such that

$$g_{ji} = \delta_{ji}, \quad \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} = 0$$

at  $P_0$ . We assume that  $N$  is sufficiently small so that there exists a positive number  $\varepsilon$  such that  $g$  satisfies in  $N$

$$|g_{ji} - \delta_{ji}| < \varepsilon, \quad |g^{ji} - \delta_{ji}| < \varepsilon, \quad \left| \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} \right| < \varepsilon.$$

Now we want to take a suitable  $C^\infty$  tensor field  $D_{ji}$ . We know that for any given tensor field  $D_{ji}$  there exists  $g(t)$  such that

$$\left(\frac{\partial g_{ji}}{\partial t}\right)_0 = D_{ji}.$$

First, we assume  $D_i^i = 0$  on  $M$ . Then

$$\left(\frac{d^2I}{dt^2}\right)_0 = \int_M \left[ (\nabla^j D^{ih}) \nabla_h D_{ji} - \frac{1}{2} (\nabla^h D^{ji}) \nabla_h D_{ji} + \frac{K}{n} D_{ji} D^{ji} \right] dV.$$

As we have

$$|(\nabla^j D^{ih} + \nabla^i D^{jh}) \nabla_h D_{ji}| = |(\nabla_a D_{ba} + \nabla_b D_{ca})(\nabla_h D_{ji}) g^{aj} g^{bi} g^{ah}|,$$

where

$$\nabla_j D_{ih} = \partial_j D_{ih} - \left\{ \begin{matrix} k \\ ji \end{matrix} \right\} D_{kh} - \left\{ \begin{matrix} k \\ jh \end{matrix} \right\} D_{ik} ,$$

we get

$$\begin{aligned} & |(\nabla^j D^{ih} + \nabla^i D^{jh})\nabla_h D_{ji}| \\ &= \left| \left( \partial_c D_{ba} + \partial_b D_{ca} - 2 \left\{ \begin{matrix} t \\ cb \end{matrix} \right\} D_{ta} - \left\{ \begin{matrix} t \\ ca \end{matrix} \right\} D_{tb} - \left\{ \begin{matrix} t \\ ba \end{matrix} \right\} D_{tc} \right) \right. \\ &\quad \left. \cdot \left( \partial_h D_{ji} - \left\{ \begin{matrix} s \\ hj \end{matrix} \right\} D_{si} - \left\{ \begin{matrix} s \\ hi \end{matrix} \right\} D_{sj} \right) g^{cj} g^{bi} g^{ah} \right| . \end{aligned}$$

Let us denote this quantity by  $F_1$ .

Define  $S_{ji}$  by

$$g^{ji} = \delta_{ji} + \epsilon S_{ji} .$$

Then  $S_{ji}$  satisfy  $|S_{ji}| < 1$ .

Assume that  $D_{ji}$  vanishes everywhere except in the interior of  $N$ , and define  $M_1$  and  $M_2$  by

$$\begin{aligned} M_1 &= \max \{ |D_{ji}(P)| ; P \in N ; i, j = 1, \dots, n \} , \\ M_2 &= \max \{ |\partial_j D_{ih}(P)| ; P \in N ; h, i, j = 1, \dots, n \} . \end{aligned}$$

Then

$$\begin{aligned} F_1 &= \sum_{a,b,c} \sum_{h,i,j} \left( \partial_c D_{ba} + \partial_b D_{ca} - 2 \left\{ \begin{matrix} t \\ cb \end{matrix} \right\} D_{ta} - \left\{ \begin{matrix} t \\ ca \end{matrix} \right\} D_{tb} - \left\{ \begin{matrix} t \\ ba \end{matrix} \right\} D_{tc} \right) \\ &\quad \cdot \left( \partial_h D_{ji} - \left\{ \begin{matrix} s \\ hj \end{matrix} \right\} D_{si} - \left\{ \begin{matrix} s \\ hi \end{matrix} \right\} D_{sj} \right) \\ &\quad \cdot (\delta_{cj} + \epsilon S_{cj})(\delta_{bi} + \epsilon S_{bi})(\delta_{ah} + \epsilon S_{ah}) \\ &\leq \left| \sum_{h,i,j} (\partial_j D_{ih} + \partial_i D_{jh}) \partial_h D_{ji} \right| + 4n^2 M_1 M_2 \epsilon + 4n^4 M_1 M_2 \epsilon + 6n^4 (M_2)^2 \epsilon \\ &\quad + (M_1 + M_2)^2 \epsilon , \end{aligned}$$

where the last term  $(M_1 + M_2)^2 \epsilon$  is a substitute for the terms containing  $\epsilon^2, \epsilon^3, \dots$ .

Next consider

$$F_2 = (\nabla^h D^{ji})\nabla_h D_{ji} .$$

For this quantity we get

$$\begin{aligned}
F_2 &= \left| \sum_{a,b,c} \sum_{h,i,j} \left( \partial_c D_{ba} - \left\{ \begin{matrix} t \\ cb \end{matrix} \right\} D_{ta} - \left\{ \begin{matrix} t \\ ca \end{matrix} \right\} D_{tb} \right) \right. \\
&\quad \cdot \left( \partial_j D_{ih} - \left\{ \begin{matrix} s \\ jj \end{matrix} \right\} D_{sh} - \left\{ \begin{matrix} s \\ jh \end{matrix} \right\} D_{si} \right) \\
&\quad \left. \cdot (\delta_{cj} + \varepsilon S_{cj})(\delta_{bi} + \varepsilon S_{bi})(\delta_{ah} + \varepsilon S_{ah}) \right| \\
&\geq \sum_{h,i,j} (\partial_j D_{ih})^2 - 4n^4 M_1 M_2 \varepsilon - 3n^4 (M_2)^2 \varepsilon - (M_1 + M_2)^2 \varepsilon,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_M \left[ (\nabla^j D^{ih}) \nabla_h D_{ji} - \frac{1}{2} (\nabla^h D^{ji}) \nabla_h D_{ji} \right] dV \\
&\leq \frac{1}{2} \int_N (F_1 - F_2) dV \\
&\leq \frac{1}{2} \int_N \left[ \left| \sum_{h,i,j} (\partial_j D_{ih} + \partial_i D_{jh}) \partial_h D_{ji} \right| - \sum_{h,i,j} (\partial_j D_{ih})^2 \right. \\
&\quad \left. + 12n^4 M_1 M_2 \varepsilon + 9n^4 (M_2)^2 \varepsilon + 2(M_1 + M_2)^2 \varepsilon \right] dV.
\end{aligned}$$

Now let us consider a tensor field  $T_{ji}$ , which vanishes everywhere except in the interior of  $N$ , such that all components are identically zero except

$$T_{12} = T_{21} = f,$$

where  $f$  is a  $C^\infty$  function. By putting

$$D_{ji} = T_{ji} - \frac{1}{n} T_{lk} g^{lk} g_{ji} = T_{ji} - \frac{2}{n} f g^{12} g_{ji},$$

we get  $D_i{}^i = 0$  and

$$|\partial_j D_{ih} - \partial_j T_{ih}| \leq \left( \frac{4}{n} |f| + \frac{2}{n} |\partial_j f| \right) \delta_{ih} \varepsilon + O(\varepsilon^2).$$

Hence

$$M_1 = (\max |f|)(1 + O(\varepsilon)), \quad M_2 \leq \left( \max |\partial_j f| + \frac{4}{n} |f| \varepsilon \right) (1 + O(\varepsilon)),$$

and we can neglect all minor terms in  $F_1$  and  $F_2$ . Moreover we can replace  $D_{ih}$  and  $\partial_j D_{ih}$  by  $T_{ih}$  and  $\partial_j T_{ih}$  respectively to obtain

$$\begin{aligned} \left(\frac{d^2I}{dt^2}\right)_0 &= \int_N \left[ \sum_{h,i,j} (\partial_j T_{ih}) \partial_h T_{ji} - \frac{1}{2} \sum_{h,i,j} (\partial_j T_{ih})^2 + \frac{K}{n} \sum_{i,j} (T_{ji})^2 \right] dV \\ &= \int_N \left[ -(\partial_3 f)^2 - \dots - (\partial_n f)^2 + \frac{2K}{n} f^2 \right] dV . \end{aligned}$$

As there exist functions  $f$  for which the last integral is negative, the index of  $I$  is positive.

Thus we have proved the main theorem.

#### 4. Index of an Einstein metric

Let  $g$  be an Einstein metric. Then the right hand side of (1.5) is a quadratic functional of the tensor field  $D_{ji}$ . For convenience this integral will be denoted by  $J(D)$ . Let  $\mathcal{D}$  be the set of tensor fields  $D$  (with components  $D_{ji}$ ) which satisfy

$$(4.1) \quad \int_M g^{ji} D_{ji} dV = 0 ,$$

$$(4.2) \quad \int_M D^{ji} D_{ji} dV = 1 .$$

We now study critical points of  $J(D)$  when  $D$  moves in  $\mathcal{D}$ .

If  $E_{ji}$  denotes an infinitesimal change of  $D_{ji}$ , we have at a critical point

$$\begin{aligned} (4.3) \quad \int_M \left[ (\nabla^j E^{ih}) \nabla_h D_{ji} + (\nabla^j D^{ih}) \nabla_h E_{ji} - (\nabla^h E^{ji}) \nabla_h D_{ji} \right. \\ \left. - (\nabla^i E_j^j) \nabla^h D_{hi} - (\nabla^i D_j^j) \nabla^h E_{hi} + (\nabla^i E_j^j) \nabla_i D_h^h \right. \\ \left. + \frac{K}{n} (2D^{ji} E_{ji} - D_j^j E_i^i) \right] dV = 0 , \end{aligned}$$

where  $E_{ji}$  satisfies

$$(4.4) \quad \int_M g^{ji} E_{ji} dV = 0 , \quad \int_M D^{ji} E_{ji} dV = 0 .$$

From (4.3) we get

$$\begin{aligned} \int_M \left[ -(\nabla_h \nabla^j D^{ih} + \nabla_h \nabla^i D^{jh}) + \nabla_h \nabla^h D^{ji} + (\nabla^k \nabla^h D_{hk}) g^{ji} + \nabla^j \nabla^i D_k^k \right. \\ \left. - (\nabla_k \nabla^k D_h^h) g^{ji} + \frac{2K}{n} D^{ji} - \frac{K}{n} D_k^k g^{ji} \right] E_{ji} dV = 0 , \end{aligned}$$

where  $E_{ji}$  is restricted only by (4.4). Hence we have

$$(4.5) \quad -\nabla_k \nabla^j D^{ik} - \nabla_k \nabla^i D^{jk} + \nabla_k \nabla^k D^{ji} + \nabla^j \nabla^i D_k{}^k \\ + \left( \nabla^t \nabla^s D_{ts} - \nabla_t \nabla^t D_s{}^s - \frac{K}{n} D_k{}^k \right) g^{ji} = C_1 g^{ji} + C_2 D^{ji},$$

where  $C_1$  and  $C_2$  are constants.

If (4.5) is transvected with  $g_{ji}$  and then integrated over  $M$ , we get  $C_1 = 0$  because of (4.1). If (4.5) is transvected with  $D_{ji}$  and then integrated over  $M$ , we get

$$C_2 = \int_M \left[ 2(\nabla^j D^{ik}) \nabla_k D_{ji} - (\nabla^k D^{ji}) \nabla_k D_{ji} - 2(\nabla_j D^{ji}) \nabla_i D_k{}^k \right. \\ \left. + (\nabla^t D_s{}^s) \nabla_t D_l{}^l - \frac{K}{n} D_l{}^l D_k{}^k \right] dV$$

because of (4.2). Hence from (1.5) it follows that

$$(4.6) \quad \left( \frac{d^2 I}{dt^2} \right)_0 = \frac{1}{2} C_2 + \frac{K}{n}.$$

Let a symmetric tensor field  $A_{ji}$  be a solution of the following system of partial differential equations with an unknown constant  $C_2$ ,

$$(4.7) \quad -\nabla^k \nabla_j A_{ik} - \nabla^k \nabla_i A_{jk} + \nabla_k \nabla^k A_{ji} + \nabla_j \nabla_i A_k{}^k \\ + \left( \nabla^t \nabla^s A_{ts} - \nabla_t \nabla^t A_s{}^s - \frac{K}{n} A_k{}^k \right) g_{ji} = C_2 A_{ji},$$

when  $C_2$  takes the eigenvalue  $a$ . If  $A_{ji}$  satisfies  $\int_M g^{ji} A_{ji} dV = 0$ , and moreover  $a < -2K/n$ , then we have  $J(A) < 0$  for this tensor field  $A_{ji}$ .

If (4.7) is transvected with  $g^{ji}$  and integrated over  $M$ , we get

$$-K \int_M A_k{}^k dV = C_2 \int_M A_k{}^k dV.$$

Hence any solution  $D_{ji}$  of (4.7) satisfies (4.1) if  $K + C_2 \neq 0$ . (4.2) is always satisfied if  $D_{ji}$  is replaced by  $D_{ji}$  multiplied by a suitable number.

Let us call each value of  $C_2$ , for which (4.7) has a solution  $D_{ji}$  satisfying (4.1), an effective eigenvalue of the system (4.7).

Define a differential operator  $L$  such that from any symmetric tensor field  $A_{ji}$  a symmetric tensor field  $L_{ji}(A)$  ( $= (L(A))_{ji}$ ) is induced by

$$(4.8) \quad L_{jt}(A) = -\nabla^k \nabla_j A_{ik} - \nabla^k \nabla_i A_{jk} + \nabla_k \nabla^k A_{ji} \\ + \nabla_j \nabla_i A_k{}^k + \left( \nabla^t \nabla^s A_{ts} - \nabla_t \nabla^t A_s{}^s - \frac{K}{n} A_k{}^k \right) g_{jt}.$$



Then we can write (4.7) in the form<sup>2</sup>:  $L_{ji}(A) = C_2 A_{ji}$ .

Let  $A_{ji}$  and  $B_{ji}$  be any  $C^\infty$  symmetric tensor fields. Then

$$\int_M L_{ji}(A) B^{ji} dV = \int_M \left[ 2(\nabla_j A_{ik}) \nabla^k B^{ji} - (\nabla_k A_{ji}) \nabla^k B^{ji} - (\nabla_i A_k{}^k) \nabla_j B^{ji} - \nabla^s A_{ts} \nabla^t B_l{}^l + \nabla^t A_s{}^s \nabla_t B_l{}^l - \frac{K}{n} A_k{}^k B_l{}^l \right] dV,$$

which will be denoted by  $(A, B)$ . If  $A_{ji}$  and  $B_{ji}$  are solutions of (4.7) when  $C_2 = a$  and  $C_2 = b$  respectively, then

$$(A, B) = a \int_M A_{ji} B^{ji} dV, \quad (B, A) = b \int_M A_{ji} B^{ji} dV.$$

As  $(A, B)$  is symmetric in  $A$  and  $B$ , we get

$$\int_M A_{ji} B^{ji} dV = 0, \quad (A, B) = 0,$$

if  $a \neq b$ .

Now let  $a_1 > \dots > a_p$  be any discrete subset of effective eigenvalues of the system of equations

$$L_{ji}(A) = C_2 A_{ji}$$

such that  $a_1 < -2K/n$ . Let  $A_{ji}^k$  be a solution of (4.7) corresponding to the eigenvalue  $a_k$  and satisfying

$$\int_M g^{ji} A_{ji}^k dV = 0.$$

As

$$\int_M A_l{}^{ji} A_{ji}^k dV = 0$$

for  $l \neq k$ ,  $A_{ji}^1, \dots, A_{ji}^p$  are linearly independent.

**Definition.** Let us say that the index of the Einstein metric under consideration is  $m$  if (4.7) has just  $m$  linearly independent solutions with effective eigenvalues less than  $-2K/n$ .

Then we get the following theorem.

**Theorem 4.1.** *If (4.7) has  $p$  effective eigenvalues  $a_1 > \dots > a_p$  such that  $a_1 < -2K/n$ , then the index of the Einstein metric under consideration is not less than  $p$ . The same holds with the index of  $I(g)$  at this Einstein point.*

<sup>2</sup>  $L$  is not an elliptic operator.

Evidently all eigenvalues of (4.7) are effective except the only possible one, namely,  $C_2 = -K$ . However, the author does not know whether  $-K$  is an eigenvalue or not.

It has been pointed out by M. Berger and D. Ebin [2], [3] that in studying the variation of  $I(g)$  we need to consider only tensor fields  $D_{ji}$  which satisfy  $\nabla^j D_{ji} = 0$ , and that a tensor field  $D_{ji} = \nabla_j X_i + \nabla_i X_j$  never changes  $I(g)$ . If we put  $A_{ji} = \nabla_j X_i + \nabla_i X_j$ , we immediately find that this is a solution of (4.7) with  $C_2 = -2K/n$ .

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